GENERATION OF SURFACE COORDINATES BY ELLIPTIC

PARTIAL DIFFERENTIAL EQUATIONS*

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The problem of generating spatial coordinates by numerical methods through carefully selected mathematical models is of current interest both in mechanics and physics. A review of various methods of coordinate generation in two and three-dimensional Euclidean spaces is available in reference 1, and reference may also be made to the proceedings of two recent conferences (references 2 and 3) and a book (ref. 4) on the topic of numerical grid generation.

In this paper the problem of generation of a desired system of coordinates in a given surface has been considered which essentially is an effort directed to the problem of grid generation in a two-dimensional non-Euclidean space. The mathematical model selected for this purpose is based on the formulae of Gauss for a surface and has been discussed by the author in earlier publications (refs. 5 - 9).

The formulae of Gauss for a surface ν=const. can be written compactly by using the summation convention on repeated lower and upper indices as

$$\underset{\sim}{r}_{,\alpha\beta} = T^{\delta}_{\alpha\beta}\, \underset{\sim}{r}_{,\delta} + \underset{\sim}{n}^{(\nu)} b_{\alpha\beta} \tag{1}$$

where $\underset{\sim}{r}=(x,y,z)$, $T^{\delta}_{\alpha\beta}$ are the surface Christoffel symbols, $\underset{\sim}{n}^{(\nu)}$ is the unit surface normal on the surface ν=const., $b_{\alpha\beta}$ are the coefficients of the second fundamental form, and a comma denotes the partial derivative with respect to the surface coordinates x^{α}(ν and the other Greek indices assuming cyclic values). Inner multiplication by $G_{\nu}g^{\alpha\beta}$ of equation (1) yields

$$D\underset{\sim}{r} + G_{\nu}(\Delta_2^{(\nu)}\, x^{\delta})\, \underset{\sim}{r}_{,\delta} = \underset{\sim}{n}^{(\nu)} R, \tag{2}$$

where

$$G_{\nu} = g_{\alpha\alpha}g_{\beta\beta} - (g_{\alpha\beta})^2, \quad \nu,\alpha,\beta \text{ cyclic},$$

$$D = G_{\nu}g^{\alpha\beta}\partial_{\alpha\beta}, \tag{3a-c}$$

$$R = G_{\nu}g^{\alpha\beta}b_{\alpha\beta} = G_{\nu}\,(k_{I}^{(\nu)} + k_{II}^{(\nu)}),$$

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and

$$\Delta_2^{(\nu)} x^\alpha = -g^{\beta\gamma} T^\alpha_{\beta\gamma}, \tag{3d}$$

is the second-order differential parameter of Beltrami. In equation (3c), $k_I^{(\nu)}$ and $k_{II}^{(\nu)}$ are the principal curvatures at a point on the surface ν=const.

Equation (2) provides three coupled quasilinear elliptic partial differential equations with the Cartesian coordiantes x,y,z as dependent variables. These equations are nonhomogenous with the right hand sides depending on the components of both the normal and the mean curvature of the surface, thus reflecting some geometrical aspects of the surface in an explicit manner.

Some of the problems listed below have been solved successfully by taking equation (2) as a basic differential model.

I. If the mathematical equation of the surface is available in the form F(x,y,z)=0, then equation (2) can be used to introduce any desired coordinate system in the surface. (For discrete x,y,z values of a surface, the form F(x,y,z)=0 can be obtained either by a least square or piecewise approximation method. Knowing F(x,y,z) = 0, one can find $k_I^{(\nu)} + k_{II}^{(\nu)}$ as a function of x,y,z. Now two options are open: In the first, one can retain $\Delta_2^{(\nu)} x^\delta$ as it appears in the equation, and in the second write $\Delta_2^{(\nu)} x^\delta = P^\delta / G_\nu$, where P^δ are arbitrarily specified functions. The second option provides a control by the user on the distribution of coordinates in the surface.

II. The proposed equations can be used to generate a new coordinate system from the data of an already given coordinate system in a surface (refs. 10 and 11).

III. If the coefficients of the first and second fundamental forms have been given, then the proposed equations can be used to generate a surface satisfying the given data (surface fitting).

IV. The proposed equations can also be used to generate surfaces in the space between two arbitrary given surfaces, thus providing 3D grids in an Euclidean space (refs. 6, 7, 12, and 13).

A number of numerical and analytical results obtained by the author and his co-workers will be presented in the seminar.

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